

# On $C^\infty$ well-posedness of hyperbolic systems with multiplicities

Claudia Garetto<sup>1</sup> · Michael Ruzhansky<sup>2</sup>

Received: 29 November 2016 / Accepted: 11 February 2017 / Published online: 25 February 2017  
© The Author(s) 2017. This article is published with open access at Springerlink.com

**Abstract** In this paper, we study first-order hyperbolic systems of any order with multiple characteristics (weakly hyperbolic) and time-dependent analytic coefficients. The main question is when the Cauchy problem for such systems is well-posed in  $C^\infty$  and in  $\mathcal{D}'$ . We prove that the analyticity of the coefficients combined with suitable hypotheses on the eigenvalues guarantees the  $C^\infty$  well-posedness of the corresponding Cauchy problem.

**Keywords** Hyperbolic equations ·  $C^\infty$  well-posedness · Analytic coefficients

**Mathematics Subject Classification** Primary 35L25 · 35L40; Secondary 46F05

## 1 Introduction

This paper is devoted to hyperbolic systems of the type

$$D_t u - A(t, D_x)u = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where  $A$  is a  $m \times m$  matrix of first-order differential or pseudo-differential operators with  $t$ -analytic entries and the eigenvalues  $\lambda_1(t, \xi), \lambda_2(t, \xi), \dots, \lambda_m(t, \xi)$  of the matrix  $A(t, \xi)$  are real. In this case, we say that the matrix  $A(t, \xi)$  is hyperbolic.

---

The first author was supported by the EPSRC First grant EP/L026422/1. The second author was supported in parts by the EPSRC grant EP/K039407/1 and by the Leverhulme Grant RPG-2014-02. No new data was collected or generated during the course of this research.

---

✉ Claudia Garetto  
c.garetto@lboro.ac.uk

Michael Ruzhansky  
m.ruzhansky@imperial.ac.uk

<sup>1</sup> Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK

<sup>2</sup> Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK

It is well known that the corresponding Cauchy problem

$$\begin{aligned} D_t u - A(t, D_x)u &= 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \\ u(0, x) &= g(x), \end{aligned} \quad (1)$$

is  $C^\infty$ -well-posed if the coefficients of the system are smooth and the eigenvalues of  $A(t, \xi)$  are distinct (so (1) is strictly hyperbolic). In this case, also large time asymptotics are well studied even allowing fast oscillations in coefficients, see, e.g., [27] (and also an extended exposition of such problems in [28]).

At the same time, if we do not assume that all the eigenvalues are distinct, much less is known even if  $A(t, \xi)$  is analytic in  $t$ . For example, if we assume that the characteristics (even  $x$ -dependent) are smooth and satisfy certain transversality relations, the  $C^\infty$ -well-posedness was shown in [21]. However, in the case of only time-dependent coefficients these transversality conditions are not satisfied.

In general, in presence of multiplicities the well-posedness is usually expected to hold in Gevrey spaces, see for instance [1–6, 16, 22, 23, 25], to mention only very few, and references therein. This happens even when the coefficients are analytic. For example, for the scalar equation

$$\partial_t^2 u - 2t \partial_t \partial_x u + t^2 \partial_x^2 u = 0$$

in one space variable, the Cauchy problem is well-posed in the Gevrey class  $\gamma^s$  for  $s < 2$  and ill-posed in  $\gamma^s$  for  $s > 2$ .

The first results of this type for  $t$ -dependent hyperbolic systems of size  $2 \times 2$  and  $3 \times 3$  have been obtained by D'Ancona, Kinoshita and Spagnolo in [7, 8]. For  $x$ -dependent  $2 \times 2$ , systems some results are also available, see, e.g., [15]. Later, the former results have been extended to any matrix size by Yuzawa in [29] and to  $(t, x)$ -dependent coefficients jointly by Kajitani and Yuzawa in [20]. In such problems, the existing techniques apply equally well for equations with coefficients (or characteristics) of lower (e.g., Hölder) regularity. More precisely, if the eigenvalues of  $A$  are of Hölder order  $\alpha \in (0, 1]$  in  $t$  and their multiplicity does not exceed  $r$ , then the Cauchy problem (1) with initial data in the Gevrey class  $\gamma^s$  have a unique solution  $u$  in  $(C^1([0, T], \gamma^s(\mathbb{R}^n)))^m$  provided that

$$1 \leq s < 1 + \frac{\alpha}{r}. \quad (2)$$

In this direction, equations with even lower (e.g., distributional) regularity have been also considered, see, e.g., [14] and also [10].

Recently, different authors have studied weakly hyperbolic scalar equations with analytic coefficients (see, for instance [18] and [13]), *but systems have not been fully investigated from this point of view*. For a discussion on the  $C^\infty$  well-posedness of hyperbolic  $2 \times 2$  systems and hyperbolic systems with non-degenerate characteristics, we refer the reader to Nishitani's recent book [24].

Here, for the first time, we consider  $m \times m$  first-order hyperbolic systems with analytic coefficients and multiple eigenvalues and we prove that under suitable conditions on the matrix  $A$ , formulated in terms of its eigenvalues, they are  $C^\infty$ -well-posed, in the sense that given initial data in  $C^\infty$  the Cauchy problem (1) have a unique solution in  $(C^1([0, T]; C^\infty(\mathbb{R}^n)))^m$ .

Thus, it is the purpose of this paper to investigate under which conditions on the matrix  $A$  the solution  $u$  does actually belong to the space  $C^1([0, T]; C^\infty(\mathbb{R}^n))^m$ . The main idea is an extension to systems of the previous works on higher-order equations with analytic coefficients and lower-order terms after a reduction to block Sylvester form.

More precisely, the analysis of this paper will consist of the following three steps:

- First, we make an observation (Theorem 2.2) that the results of Yuzawa [29], and Kajitani and Yuzawa [20], can be extended to produce the existence of some (ultradistributional) solution to the Cauchy problem (1). It is then our task to improve its regularity to  $C^\infty$  or to  $\mathcal{D}'$  depending on the regularity of the Cauchy data. This step is done in Sect. 2.1.
- Second, we consider matrices  $A(t, D_x)$  in Sylvester form and prove (in Theorem 2.5) that in this case the Cauchy problem (1) is well-posed in  $C^\infty$ . This step is done in Sect. 2.2.
- Third, we extend the above to any weakly hyperbolic matrix  $A$  or, in other words, we show that we can drop the assumption of Sylvester form for the matrix  $A$ . This is done by transforming a general  $m \times m$  system

$$D_t - A(t, D_x)$$

into the  $m^2 \times m^2$  block Sylvester system, which is a key idea of the paper, so that we could use the established result in that case. This extended system will be still hyperbolic (in fact, the principal part will have the same eigenvalues), but such reduction will (unfortunately) produce some lower-order terms. Therefore, we carry out a careful analysis of the appearing matrix of the lower-order terms by considering the suitable Kovalevskian and hyperbolic energies in different frequency domains. This will yield the desired  $C^\infty$ -well-posedness as well as the distributional well-posedness for the original Cauchy problem (1) in Theorem 3.3. This analysis will be carried out in Sects. 3 and 4.

In Sect. 3.1 we illustrate the appearing Levi-type conditions in the example of  $2 \times 2$  systems. We also note that the obtained conditions can be expressed entirely in terms of the coefficients of the matrix  $A(t, x)$  (rather than its eigenvalues) and are, therefore, computable. We refer to [18] and to [13] for the discussions of such expressions.

Finally, we note that in problems concerning systems, it is often important whether the system can be diagonalised or whether it contains Jordan blocks, see, e.g., [21] or [15], for some respective results and further references. However, this is not an issue for the present paper since we are able to obtain the well-posedness results avoiding such assumptions. We also note that ideas similar to those in this paper can be also applied in other situations for less regular coefficients, see, e.g., [14] and [26].

## 2 Preliminary results

In this section, we discuss several preliminary results needed for our analysis. First, we make an observation that the results of Yuzawa [29], and Kajitani and Yuzawa [20], can be extended to produce the existence of an ultradistributional solution, thus enabling our further reductions. Then, we look at systems in the Sylvester form.

### 2.1 Ultradistributional well-posedness

For convenience of the reader we recall Yuzawa's well-posedness result proven in [29]. We begin by introducing for  $\rho > 0$  and  $s > 1$ , the space  $H_{\Lambda(\rho, s)}^l$  of all  $f \in L^2(\mathbb{R}^n)$  such that

$$\langle \xi \rangle^l e^{\Lambda(\rho, s)} \widehat{f}(\xi) \in L^2(\mathbb{R}_\xi^n),$$

where  $\Lambda(\rho, s) = \rho \langle \xi \rangle^{\frac{1}{s}}$ . Let now the coefficients of the matrix  $A$  be of class  $C^\alpha$  and let  $s$  be as in (2). Theorem 1.1 in [29] states that if the initial data  $g$  have entries in  $H_{\Lambda(T, s)}^l$ ,

then the Cauchy problem (1) has a unique solution  $u(t, x)$  such that  $e^{(T-t)\langle D_x \rangle^{\frac{1}{s}}} u(t, x) \in (C([0, T]; H^l))^m \cap (C^1([0, T]; H^{l-1}))^m$ , for  $t \in [0, T]$  and  $x \in \mathbb{R}^n$ . From Lemma 1.2 in [19] by Kajitani, one has that for any  $f \in \gamma_c^s(\mathbb{R}^n)$  and  $l \in \mathbb{R}$  there exists  $\rho > 0$  (depending on  $f$ ) such that  $f \in H_{\Lambda(\rho, s)}^l$  and conversely, if  $f$  is a compactly supported element of some  $H_{\Lambda(\rho, s)}^l$ , then it is a compactly supported Gevrey function of order  $s$ . It then follows that the previous well-posedness results in  $H_{\Lambda(\rho, s)}^l$  spaces can be formulated in Gevrey classes. More precisely, Theorem 1.2 in [29] states that given initial data with entries in  $\gamma_c^s(\mathbb{R}^n)$  for  $s$  as in (2), there exists a unique solution  $u \in C^1([0, T]; \gamma^s(\mathbb{R}^n))^m$  of the Cauchy problem (1). For the advantage of the reader, we recall that hyperbolic equations and systems possess the finite speed of propagation property. This ensures that if the initial data are compactly supported, then the solution  $u$  is compactly supported with respect to  $x$  ( $u \in C^1([0, T]; \gamma_c^s(\mathbb{R}^n))^m$ ) and that Theorem 1.2 holds for non-compactly supported initial data as well.

Note that the characterisation of Gevrey functions via weighted Sobolev spaces can be extended to Gevrey Beurling ultradistributions. We recall that  $f \in C^\infty(\mathbb{R}^n)$  belongs to the Beurling Gevrey class  $\gamma^{(s)}(\mathbb{R}^n)$  if for every compact set  $K \subset \mathbb{R}^n$  and for every constant  $A > 0$  there exists a constant  $C_A > 0$  such that for all  $\alpha \in \mathbb{N}_0^n$  the estimate

$$|\partial^\alpha f(x)| \leq C_A A^{|\alpha|} (\alpha!)^s$$

holds uniformly in  $x \in K$ . The space  $\mathcal{D}'_{(s)}(\mathbb{R}^n)$  of Gevrey Beurling ultradistributions is defined as the dual of  $\gamma_c^{(s)}(\mathbb{R}^n)$ , while the space of  $\mathcal{E}'_{(s)}(\mathbb{R}^n)$  of compactly supported Gevrey Beurling ultradistributions is the dual of  $\gamma^{(s)}(\mathbb{R}^n)$ . In analogy to Gevrey classes, one has that a real analytic functional  $v$  belongs to  $\mathcal{E}'_s(\mathbb{R}^n)$  if and only if for any  $\nu > 0$  there exists  $C_\nu > 0$  such that

$$|\widehat{v}(\xi)| \leq C_\nu e^{\nu \langle \xi \rangle^{\frac{1}{s}}}$$

for all  $\xi \in \mathbb{R}^n$ , and similarly,  $v \in \mathcal{E}'_{(s)}(\mathbb{R}^n)$  if and only if there exist  $\nu > 0$  and  $C > 0$  such that

$$|\widehat{v}(\xi)| \leq C e^{\nu \langle \xi \rangle^{\frac{1}{s}}}$$

for all  $\xi \in \mathbb{R}^n$  (see Proposition 13 in [11]). Combining these observations with Kajitani and Yuzawa's method in [29] and [20], one can easily extend Lemma 1.2 in [19] and deduce the corresponding ultradistributional well-posedness results. More precisely, we have the following lemma and well-posedness theorems.

**Lemma 2.1** (i) For any  $v \in \mathcal{E}'_{(s)}(\mathbb{R}^n)$  and  $l \in \mathbb{R}$  there exists  $\rho > 0$  such that  $v \in H_{-\Lambda(\rho, s)}^l$ .  
(ii) If  $v \in H_{-\Lambda(\rho, s)}^l$  is compactly supported then  $v \in \mathcal{E}'_{(s)}(\mathbb{R}^n)$ .

*Proof* (i) From the Fourier characterisation of ultradistributions, we have that there exist constants  $c > 0$  and  $\rho > 0$  such that

$$|\widehat{v}(\xi)| \leq c e^{\rho \langle \xi \rangle^{\frac{1}{s}}},$$

for all  $\xi \in \mathbb{R}^n$ . It follows that

$$\langle \xi \rangle^l e^{-(\rho+1)\langle \xi \rangle^{\frac{1}{s}}} |\widehat{v}(\xi)| \leq c \langle \xi \rangle^l e^{-\langle \xi \rangle^{\frac{1}{s}}},$$

where the right-hand side is clearly an element of  $L^2$ . Thus,  $v \in H_{-\Lambda(\rho, s)}^l$ .

(ii) Let now  $A(\mathbb{R}^n)$  be the set of analytic functions and  $H_{-\Lambda(\rho,s)}^l$  be the set of all functionals  $v$  on  $A(\mathbb{R}^n)$  such that

$$\langle \xi \rangle^l e^{-\Lambda(\rho,s)} \widehat{v}(\xi) \in L^2(\mathbb{R}_\xi^n). \quad (3)$$

Assuming that  $v$  is compactly supported, we know that  $\widehat{v}$  is an analytic function satisfying an estimate of the type

$$|\widehat{v}(\xi)| \leq c \langle \xi \rangle^N, \quad (4)$$

for some  $c > 0$  and  $N \in \mathbb{N}_0$ . Since we can write (3) as

$$\langle \xi \rangle^l e^{-\Lambda(\rho,s)} \widehat{v}(\xi) = g(\xi),$$

where  $g \in L^2(\mathbb{R}^n)$ , by using (4) we conclude that  $|g(\xi)| \leq c_1 e^{-\rho_1 \langle \xi \rangle^{\frac{1}{s}}}$  for some  $c_1, \rho_1 > 0$ . Hence, it follows that

$$|\widehat{v}(\xi)| \leq c' e^{\rho \langle \xi \rangle^{\frac{1}{s}}}.$$

This proves that  $v$  is an ultradistribution in  $\mathcal{E}'_{(s)}(\mathbb{R}^n)$ .  $\square$

We can now recall the precise form of Kajitani–Yuzawa result described earlier.

**Theorem 2.2** *Let the coefficients of the matrix  $A$  be of class  $C^\alpha$  and let  $A$  have real eigenvalues which do not exceed the multiplicity  $r$  and let*

$$1 \leq s < 1 + \frac{\alpha}{r}.$$

*Then, for any initial data  $g$  with entries in  $H_{-\Lambda(T,s)}^l$  the Cauchy problem (1) has a unique solution  $u(t, x)$  such that*

$$e^{-(T-t)\langle D_x \rangle^{\frac{1}{s}}} u(t, x) \in (C([0, T]; H^l))^m \cap (C^1([0, T]; H^{l-1}))^m,$$

*for  $t \in [0, T]$  and  $x \in \mathbb{R}^n$ .*

As a consequence of Lemma 2.1 and Theorem 2.2, we obtain the following ultradistributional well-posedness result which will be the starting point for our analysis.

**Theorem 2.3** *Under the hypotheses of Theorem 2.2 for any initial data  $g$  with entries in  $\mathcal{E}'_{(s)}(\mathbb{R}^n)$ , the Cauchy problem (1) has a unique ultradistributional solution  $u \in C^1([0, T]; \mathcal{D}'_{(s)}(\mathbb{R}^n))^m$ .*

We now turn to a preliminary setting of Sylvester matrices.

## 2.2 Systems in Sylvester form

From now on, we concentrate on the Cauchy problem (1)

$$\begin{aligned} D_t u - A(t, D_x) u &= 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \\ u(0, x) &= g(x), \end{aligned}$$

when the entries of the matrix  $A$  are analytic in  $t$ . By applying Theorem 2.3, we already know that if we take initial data in  $(C_c^\infty(\mathbb{R}^n))^m$ , then a solution  $u$  exists in  $C^1([0, T]; \mathcal{D}'_{(s)}(\mathbb{R}^n))^m$ .

First, we briefly collect some preliminaries, for more details we refer the reader to [13, 18].

Thus, here we assume that  $A(t, \xi)$ , the matrix of the principal part of the operator  $D_t u - A(t, D_x)$ , is a matrix of first-order *pseudo-differential operators* of Sylvester type (we will show in the next section that this assumption is not restrictive). It means that we can write  $A(t, \xi)$  as  $\langle \xi \rangle A_0(t, \xi)$ , where

$$A_0(t, \xi) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ h_m & h_{m-1} & \dots & \dots & h_1 \end{pmatrix}, \quad (5)$$

for some  $h_j$ ,  $j = 1, \dots, m$ , symbols of order 0 analytic in  $t$ . The eigenvalues of  $A_0(t, \xi)$  are exactly the eigenvalues of  $A(t, \xi)$  scaled by a factor  $\langle \xi \rangle^{-1}$ , i.e.,  $\langle \xi \rangle^{-1} \lambda_j(t, \xi)$ ,  $j = 1, \dots, m$ . Hence, they are symbols of order 0 in  $\xi$  analytic with respect to  $t$ .

Let us now fix  $t$  and  $\xi$  and treat  $A_0$  as a matrix with constant entries. Since  $A_0$  is hyperbolic, we can construct a real symmetric semi-positive definite  $m \times m$  matrix  $Q$  such that

$$QA_0 - A_0^* Q = 0 \quad (6)$$

and

$$\det Q = \prod_{1 \leq k < j \leq m} \langle \xi \rangle^{-2} (\lambda_j - \lambda_k)^2.$$

The matrix  $Q$  is called the standard symmetriser of  $A_0$ . Its entries are fixed polynomials functions of  $h_1, \dots, h_m$  (or, equivalently, they can be expressed via the eigenvalues of  $A_0$ ), and it is weakly positive definite if and only if  $A_0$  is weakly hyperbolic (see [17]).

Let  $Q_j$  be the principal  $j \times j$  minor of  $Q$  obtained by removing the first  $m - j$  rows and columns of  $Q$  and let  $\Delta_j$  be its determinant. When  $j = m$ , we use the notations  $Q$  and  $\Delta$  instead of  $Q_m$  and  $\Delta_m$ . The following proposition shows how the hyperbolicity of  $A_0$  (or equivalently of  $A$ ) can be seen at the level of the symmetriser  $Q$  and of its minors (see [17]).

**Proposition 2.4** (i)  $A$  is strictly hyperbolic if and only if  $\Delta_j > 0$  for all  $j = 1, \dots, m$ .  
(ii)  $A$  is weakly hyperbolic if and only if there exists  $r < m$  such that

$$\Delta = \Delta_{m-1} = \dots = \Delta_{r+1} = 0$$

and  $\Delta_r > 0, \dots, \Delta_1 > 0$ . (In this case there are exactly  $r$  distinct roots).

Clearly, when  $t$  and  $\xi$  vary in their domains, respectively,  $\Delta_r$  becomes a symbol  $\Delta_r(t, \xi)$  homogeneous of degree 0 in  $\xi$  and analytic in  $t$ . When  $\Delta_r$  is not identically zero, one can define the function

$$\tilde{\Delta}_r(t, \xi) = \Delta_r(t, \xi) + \frac{(\partial_t \Delta_r(t, \xi))^2}{\Delta_r(t, \xi)},$$

which is as well a symbol of order 0 in  $\xi$  and analytic in  $t$ . Note that if  $t \mapsto \Delta(t, \xi)$  vanishes of order  $2k$  at a point  $t'$ , then  $t \mapsto \tilde{\Delta}(t, \xi)$  vanishes of order  $2k - 2$  at  $t'$ .

In analogy with the scalar equation case treated in [13], the energy estimate that we will use for the system  $D_t u - A(t, D_x)u = 0$ , when  $A$  is in Sylvester form, will make use of the quotient

$$\langle \partial_t QV, V \rangle / \langle QV, V \rangle. \quad (7)$$

As already observed in [13], estimating the quotient  $\langle \partial_t QV, V \rangle / \langle QV, V \rangle$  is equivalent to estimating the roots of the generalised Hamilton-Cayley polynomial

$$\det(\tau Q - \partial_t Q) = \sum_{j=0}^m d_j(t) \tau^{m-j} \quad (8)$$

of  $Q$  and  $\partial_t Q$ , where  $d_0 = \det Q$ ,  $d_1 = -\partial_t(\det Q)$ ,  $d_m = (-1)^m \det(\partial_t Q)$  and, if  $m \geq 2$ ,  $d_2 = \frac{1}{2} \text{trace}(\partial_t Q \partial_t(Q^{\text{co}}))$ , where  $Q^{\text{co}}$  is the cofactor matrix of  $Q$ . From the identity

$$\tau_1^2 + \cdots + \tau_m^2 = \left( \frac{d_1}{d_0} \right)^2 - 2 \frac{d_2}{d_0},$$

valid for the roots  $\tau_j$ ,  $j = 1, \dots, m$ , of the generalised Hamilton-Cayley polynomial, we easily see that  $d_2$  is crucial when estimating (7). Let

$$\psi(t, \xi) := d_2(t, \xi) = \frac{1}{2} \text{trace}(\partial_t Q \partial_t(Q^{\text{co}})),$$

and we call  $\psi$  the *check function* of  $Q$ .

For the moment, we work under the following set (H) of hypotheses:

- (i)  $A$  is a matrix of pseudo-differential operators of order 1,
- (ii)  $A$  is in Sylvester form.

We can now state our preliminary well-posedness result for the Cauchy problem (1). This result is obtained from Theorem 2.2 in [13] where the well-posedness in the scalar case is obtained by reduction to a first-order pseudo-differential system with principal part in Sylvester form. Note that for technical reasons we will work on slightly bigger open interval  $(\delta, T' + \delta)$  containing  $[0, T]$ .

**Theorem 2.5** *Let  $D_t - A(t, D_x)$  be the matrix operator in (1) under the hypotheses (H). Let the entries of  $A(t, D_x)$  be analytic in  $t \in (\delta, T' + \delta)$  and let the matrix  $A(t, \xi)$  be (weakly) hyperbolic. Let  $Q(t, \xi) = \{q_{ij}(t, \xi)\}_{i,j=1}^m$  be the symmetriser of the matrix  $A_0(t, \xi) = \langle \xi \rangle^{-1} A(t, \xi)$ ,  $\Delta$  its determinant and  $\psi(t, \xi)$  its check function. Let  $\Delta(\cdot, \xi) \neq 0$  in  $(\delta, T' + \delta)$  for all  $\xi$  with  $|\xi| \geq 1$  and let  $[0, T] \subset (\delta, T' + \delta)$ . Assume that there exists a constant  $C_1 > 0$  such that*

$$|\psi(t, \xi)| \leq C_1 \tilde{\Delta}(t, \xi) \quad (9)$$

*holds for all  $t \in [0, T]$  and  $|\xi| \geq 1$ . Then the Cauchy problem*

$$\begin{aligned} D_t u - A(t, D_x) u &= 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \\ u(0, x) &= g(x), \end{aligned} \quad (10)$$

*is  $C^\infty$  well-posed, in the sense that given  $g \in (C^\infty(\mathbb{R}^n))^m$  there exists a unique solution  $u$  in  $C^\infty([0, T], C^\infty(\mathbb{R}^n))^m$ , and it is also well-posed in  $\mathcal{D}'(\mathbb{R}^n)$ , i.e., for any  $g \in (\mathcal{D}'(\mathbb{R}^n))^m$  there exists a unique solution  $u \in C^\infty([0, T], \mathcal{D}'(\mathbb{R}^n))^m$ .*

For simplicity, we will refer to the well-posedness above as  $C^\infty$  well-posedness and distributional well-posedness in the interval  $[0, T]$ . Note that by the energy estimates we obtain first that the solution is  $C^1$  with respect to  $t \in [0, T]$  and then, by iterated differentiation in the original system, we conclude that the dependence in  $t$  is actually  $C^\infty$ .

Our next aim is to extend the theorem above to any weakly hyperbolic matrix  $A$ , or in other words to drop the assumption of Sylvester form for the matrix  $A$ . This will be done by reducing a general system

$$D_t - A(t, D_x)$$

into block Sylvester form. Unfortunately, this will produce some lower-order terms and therefore a careful analysis of the new matrix  $B$  of the lower-order terms will be needed to achieve  $C^\infty$  and distributional well-posedness. This will be done in the next sections.

### 3 Main result

We perform a reduction to block Sylvester form of the system in (1) by following the ideas of D'Ancona and Spagnolo in [9]. We begin by considering the cofactor matrix  $L(t, \tau, \xi)$  of  $(\tau I - A(t, \xi))^T$  where  $I$  is the  $m \times m$  identity matrix. By applying the corresponding operator  $L(t, D_t, D_x)$  to (1) we transform the system

$$D_t u - A(t, D_x)u = 0$$

into

$$\mu(t, D_t, D_x)Iu - C(t, D_t, D_x)u = 0, \quad (11)$$

where  $\mu(t, \tau, \xi) = \det(\tau I - A(t, \xi))$  and  $C(t, D_t, D_x)$  is the matrix of lower-order terms (differential operators of order  $m - 1$ ). More precisely,  $\mu(t, D_t, D_x)$  is an operator of the form

$$\mu(t, D_t, D_x) = D_t^m + \sum_{h=0}^{m-1} b_{m-h}(t, D_x)D_t^h,$$

with  $b_{m-h}(t, \xi)$  a homogeneous polynomial of order  $m - h$ .

We now transform this set of scalar equations of order  $m$  into a first-order system of size  $m^2 \times m^2$  of pseudo-differential equations, by setting

$$U = \{D_t^{j-1} \langle D_x \rangle^{m-j} u\}_{j=1,2,\dots,m},$$

where  $\langle D_x \rangle$  is the pseudo-differential operator with symbol  $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ . We can therefore write (11) in the form

$$D_t U - \mathcal{A}(t, D_x)U + \mathcal{L}(t, D_x)U = 0, \quad (12)$$

where  $\mathcal{A}$  is a  $m^2 \times m^2$  matrix made of  $m$  identical blocks of the type

$$\langle D_x \rangle \cdot \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_m(t, D_x) \langle D_x \rangle^m - b_{m-1}(t, D_x) \langle D_x \rangle^{-m+1} & \cdots & \cdots & -b_1(t, D_x) \langle D_x \rangle^{-1} \end{pmatrix}, \quad (13)$$



with  $b_j(t, D_x)$  a pseudo-differential operator of order  $j$ ,  $j = 1, \dots, m$ , analytic in  $t$ , and the matrix  $\mathcal{L}$  of the lower-order terms is made of  $m$  blocks of size  $m \times m^2$  of the type

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{i,1}(t, D_x) & l_{i,2}(t, D_x) & \cdots & \cdots & l_{i,m^2-1}(t, D_x) & l_{i,m^2}(t, D_x) \end{pmatrix},$$

with  $i = 1, \dots, m$ . Note that the operators  $l_{i,j}$ ,  $j = 1, \dots, m^2$ , are all of order 0 in  $\xi$ . Hence, by construction the matrices  $\mathcal{A}$  and  $\mathcal{L}$  are made by pseudo-differential operators of order 1 and 0, respectively. Concluding, the Cauchy problem (1) has been now transformed into

$$\begin{aligned} D_t U - \mathcal{A}(t, D_x) U + \mathcal{L}(t, D_x) U &= 0, \\ U_{t=0} &= \{D_t^{j-1} \langle D_x \rangle^{m-j} u(0, x)\}_{j=1,2,\dots,m}. \end{aligned} \quad (14)$$

This is a Cauchy problem of first-order pseudo-differential equations with principal part in block Sylvester form. The size of the system has increased from  $m \times m$  to  $m^2 \times m^2$ , but the system is still hyperbolic, since the eigenvalues of any block of  $\mathcal{A}(t, \xi)$  are the eigenvalues of the matrix  $\langle \xi \rangle^{-1} A(t, \xi)$ .

We now want to analyse the matrix  $\mathcal{L}$  in more detail and study its relationship with the principal matrix  $A$ . For this purpose, we observe that by definition of the operator  $L(t, D_t, D_x)$  we have that

$$L(t, D_t, D_x) = \sum_{h=0}^{m-1} A_h(t, D_x) D_t^{m-1-h},$$

where

$$A_h(t, D_x) = (-1)^m \sum_{h'=0}^h \sigma_{h'}^{(m)}(\lambda) A^{h-h'}(t, D_x), \quad (15)$$

with  $\lambda = (\lambda_1, \dots, \lambda_m)$ ,

$$\sigma_{h'}^{(m)}(\lambda) = (-1)^{h'} \sum_{1 \leq i_1 < \dots < i_{h'} \leq m} \lambda_{i_1} \dots \lambda_{i_{h'}}$$

and  $\sigma_0^{(m)}(\lambda) = 1$ . We can now prove the following linear algebra lemma.

**Lemma 3.1** *The entries of the matrix  $\mathcal{L}$  of the lower-order terms are of the type*

$$\langle \xi \rangle^{-1} \sum_{k=1}^{m-1} c_{i(k), j(k)}(t) D_t^k a_{i(k), j(k)}(t, \xi),$$

where  $1 \leq i(k), j(k) \leq m$  and  $c_{i(k), j(k)}$  is a bounded function in  $t$ .

*Proof* We apply the operator  $L(t, D_t, D_x)$  to  $D_t I - A(t, D_x)$ . By direct computations and by formula (15), we have that

$$\begin{aligned} L(t, D_t, D_x)(D_t I - A(t, D_x)) &= \sum_{h=0}^{m-1} A_h(t, D_x) D_t^{m-1-h} \\ &\quad - \sum_{h=0}^{m-1} A_h(t, D_x) \sum_{q=0}^{m-1-h} \binom{m-1-h}{q} D_t^q A(t, D_x) D_t^{m-1-h-q} \end{aligned} \quad (16)$$

By now writing the last term in (16) as  $-X - Y$ , where

$$X = \sum_{h=0}^{m-1} A_h(t, D_x) A(t, D_x) D_t^{m-1-h}$$

and

$$Y = \sum_{h=0}^{m-1} A_h(t, D_x) \sum_{q=1}^{m-1-h} \binom{m-1-h}{q} D_t^q A(t, D_x) D_t^{m-1-h-q}$$

we easily see that  $\sum_{h=0}^{m-1} A_h(t, D_x) D_t^{m-h} - X = \mu(t, D_t, D_x)$ , i.e., the principal part of the operator  $L(t, D_t, D_x)(D_t I - A(t, D_x))$ , while the lower-order terms  $C(t, D_t, D_x)$  are given by  $-Y$ . Hence,

$$C(t, D_t, D_x) = \sum_{h=0}^{m-1} A_h(t, D_x) \sum_{q=1}^{m-1-h} \binom{m-1-h}{q} D_t^q A(t, D_x) D_t^{m-1-h-q}.$$

Note that  $A_h$  contains only powers of the operator  $A$  up to order  $h$  and therefore  $C$  contains powers of  $A$  up to order  $m-1$  and derivatives of  $A$  from order 1 to order  $m-1$ . Passing now to the reduction to a first-order system of size  $m^2 \times m^2$  of pseudo-differential operators, we easily see that the entries of the matrix  $\mathcal{L}$  in (12) are obtained by the matrix  $C$  and therefore from  $A_h D_t^q A$  suitably reduced to order 0, i.e.,  $\langle \xi \rangle^{-h} A_h(t, \xi) D_t^q A(t, \xi) \langle \xi \rangle^{-1}$ . Since  $\langle \xi \rangle^{-h} A_h(t, \xi)$  is bounded with respect to  $t$  and  $\xi$  and  $1 \leq q \leq m-1$ , we conclude that the entries of the matrix  $\mathcal{L}$  are of the desired type.  $\square$

The representation formula in Lemma 3.1 implies the following estimate.

**Proposition 3.2** *The matrix  $\mathcal{L}$  is bounded by the derivatives of the matrix  $A_0 = \langle \xi \rangle^{-1} A$  up to order  $m-1$ , i.e., there exists a constant  $c > 0$  such that*

$$\|\mathcal{L}(t, \xi)\| \leq c \max_{k=1, \dots, m-1} \|D_t^k A_0(t, \xi)\|, \quad (17)$$

for all  $t \in [0, T]$  and  $\xi \in \mathbb{R}^n$ , where  $\|\cdot\|$  denotes the standard matrix norm.

We can now state our main result, which extends Theorem 2.5 to a general hyperbolic matrix  $A$ .

**Theorem 3.3** *Let  $D_t - A(t, D_x)$  be the matrix operator in (1). Let the entries of  $A(t, D_x)$  be analytic in  $t \in (\delta, T' + \delta)$  and let the matrix  $A(t, \xi)$  be (weakly) hyperbolic. Let  $Q(t, \xi) = \{q_{ij}(t, \xi)\}_{i,j=1}^m$  be the symmetriser of the matrix  $A_0(t, \xi) = \langle \xi \rangle^{-1} A(t, \xi)$ ,  $\Delta$  its determinant and  $\psi(t, \xi)$  its check function. Let  $\Delta(\cdot, \xi) \not\equiv 0$  in  $(\delta, T' + \delta)$  for all  $\xi$  with  $|\xi| \geq 1$  and let  $[0, T] \subset (\delta, T' + \delta)$ . Assume that there exists a constant  $C > 0$  such that*

$$|\psi(t, \xi)| \leq C \tilde{\Delta}(t, \xi) \quad (18)$$

and

$$\max_{k=1, \dots, m-1} \|\partial_t^k A_0(t, \xi)\| \leq C(\Delta(t, \xi) + \partial_t \Delta(t, \xi)) \quad (19)$$

for all  $t \in [0, T]$  and  $|\xi| \geq 1$ . Then the Cauchy problem

$$\begin{aligned} D_t u - A(t, D_x) u &= 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \\ u(0, x) &= g(x), \end{aligned} \quad (20)$$

is  $C^\infty$  well-posed and distributionally well-posed in  $[0, T]$ .

Before proceeding with the energy estimate which will allow us to prove Theorem 3.3 we focus on the case  $m = 2$ . The following explanatory example will help the reader to better understand the meaning of the hypotheses (18) and (19).

### 3.1 Example: the case $m = 2$

We recall that if  $\lambda_1, \lambda_2$  are the eigenvalues of  $A$  then

$$Q(t, \xi) = \begin{pmatrix} \langle \xi \rangle^{-2}(\lambda_1^2 + \lambda_2^2)(t, \xi) & -\langle \xi \rangle^{-1}(\lambda_1 + \lambda_2)(t, \xi) \\ -\langle \xi \rangle^{-1}(\lambda_1 + \lambda_2)(t, \xi) & 2 \end{pmatrix},$$

with

$$\Delta = \langle \xi \rangle^{-2}(\lambda_1 - \lambda_2)^2(t, \xi)$$

and

$$\tilde{\Delta} = \langle \xi \rangle^{-2}(\lambda_1 - \lambda_2)^2(t, \xi) + 2\langle \xi \rangle^{-2}(\partial_t \lambda_1 - \partial_t \lambda_2)^2(t, \xi),$$

and

$$\psi(t, \xi) = \frac{1}{2} \text{trace}(\partial_t Q \partial_t(Q^{\text{co}}))(t, \xi) = -\langle \xi \rangle^2(\partial_t \lambda_1 + \partial_t \lambda_2)^2(t, \xi).$$

It follows that in this case the hypothesis (18) looks like

$$(\partial_t \lambda_1 + \partial_t \lambda_2)^2(t, \xi) \leq C((\lambda_1 - \lambda_2)^2(t, \xi) + (\partial_t \lambda_1 - \partial_t \lambda_2)^2(t, \xi))$$

and (19) is given by

$$\|\partial_t A_0(t, \xi)\| \leq C\langle \xi \rangle^{-2}((\lambda_1 - \lambda_2)^2(t, \xi) + |(\lambda_1 - \lambda_2)(t, \xi)(\partial_t \lambda_1 - \partial_t \lambda_2)(t, \xi)|).$$

Note that when the matrix  $A$  is already in Sylvester form, the formulation of the hypotheses (18) and (19) is simplified and sometimes trivial. For instance, when

$$A(t, \xi) = \xi \begin{pmatrix} 0 & 1 \\ a^2(t) & 0 \end{pmatrix},$$

$\xi \in \mathbb{R}$ , both the hypotheses (18) and (19) are trivially satisfied. Indeed,  $\lambda_1(t, \xi) = -|a(t, \xi)|$  and  $\lambda_2(t, \xi) = |a(t, \xi)|$ . This implies (18) because  $\psi(t, \xi) \equiv 0$  and (19) becomes

$$|2a(t)a'(t)| \leq C(4a^2(t) + 4|a(t)a'(t)|),$$

which is trivially true.

Some results on the  $C^\infty$  well-posedness for  $2 \times 2$  hyperbolic systems with analytic coefficients have been obtained in [8]. Although not directly comparable, both types of conditions have their advantages from different points of view (for instance the formulation for any matrix size in our case).

## 4 Proof of the main theorem

The proof of Theorem 3.3 is partly based on the analogous result for scalar equations in [13] to which we will refer for the complete details of some steps of the proof. This is due to the

reduction to block Sylvester form explained in the previous section which allows to define the block diagonal  $m^2 \times m^2$ -symmetriser

$$Q = \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & Q \end{pmatrix},$$

where  $Q$  is the symmetriser of the matrix  $A_0 = \langle \xi \rangle^{-1} A$ . Since the reduction to block Sylvester form transforms the original system

$$D_t u - A(t, D_x)u = 0$$

into the system

$$D_t U - \mathcal{A}(t, D_x)U + \mathcal{L}(t, D_x)U = 0,$$

with  $\mathcal{A}$  in the block Sylvester form, our proof will need to take care of the lower-order terms in  $\mathcal{L}$  which do not enter into  $\mathcal{A}$ . This will be done by using the Levi conditions introduced in [13] and in particular by referring to Remark 4.8 in [13].

We begin by recalling some technical lemmas which have been proved in [13] which will be useful for our analysis of systems as well.

**Lemma 4.1** *Let  $Q(t, \xi)$  be the symmetriser of the weakly hyperbolic matrix  $A(t, \xi)$  defined above. Then, there exist two positive constants  $c_1$  and  $c_2$  such that*

$$c_1 \det Q(t, \xi) |V|^2 \leq \langle Q(t, \xi)V, V \rangle \leq c_2 |V|^2$$

*holds for all  $t \in [0, T]$ ,  $\xi \in \mathbb{R}^n$  and  $V \in \mathbb{C}^m$ .*

**Lemma 4.2** *Let  $Q(t, \xi)$  be the symmetriser of the matrix  $A(t, \xi)$ . Let  $\Delta(t, \xi) = \det Q(t, \xi)$ ,  $\widetilde{\Delta}(t, \xi) = \Delta(t, \xi) + (\partial_t \Delta(t, \xi))^2 / \Delta(t, \xi)$ ,  $\psi(t, \xi)$  the check function of  $Q(t, \xi)$ . Let  $I$  be a closed interval of  $\mathbb{R}$ . Then,*

$$\sqrt{\frac{\Delta(t, \xi)}{\widetilde{\Delta}(t, \xi)}} \frac{\langle \partial_t Q(t, \xi)V, V \rangle}{\langle Q(t, \xi)V, V \rangle} \in L^\infty(I \times \mathbb{R}^n \times \mathbb{C}^m \setminus 0) \quad (21)$$

*if and only if*

$$\frac{\psi(t, \xi)}{\widetilde{\Delta}(t, \xi)} \in L^\infty(I \times \mathbb{R}^n). \quad (22)$$

**Remark 4.3** It is clear that Lemma 4.1 and Lemma 4.2 are valid also for the block diagonal matrix  $\mathcal{A}$  and the corresponding symmetriser  $Q$  as defined at the beginning of this section.

**Lemma 4.4** *Let  $\Delta(t, \xi)$  be the determinant of  $Q(t, \xi)$  defined as above. Suppose that  $\Delta(t, \xi) \neq 0$ . Then,*

- (i) *there exists  $X \subset \mathbb{S}^{n-1}$  such that  $\Delta(t, \xi) \neq 0$  in  $(\delta, T' + \delta)$  for any  $\xi \in X$  and the set  $\mathbb{S}^{n-1} \setminus X$  is negligible with respect to the Hausdorff  $(n-1)$ -measure;*
- (ii) *for any  $[0, T] \subset (\delta, T' + \delta)$  there exist  $c_1, c_2 > 0$  and  $p, q \in \mathbb{N}_0$  such that for any  $\xi \in X$  and any  $\varepsilon \in (0, e^{-1}]$  there exists  $A_{\xi, \varepsilon} \subset [a, b]$  such that:*

- $A_{\xi, \varepsilon}$  is a union of at most  $p$  disjoint intervals,
- $\text{meas}(A_{\xi, \varepsilon}) \leq \varepsilon$ ,

$$- \min_{t \in [0, T] \setminus A_{\xi, \varepsilon}} \Delta(t, \xi) \geq c_1 \varepsilon^{2q} \|\Delta(\cdot, \xi)\|_{L^\infty([0, T])},$$

$$\int_{t \in [0, T] \setminus A_{\xi, \varepsilon}} \frac{|\partial_t \Delta(t, \xi)|}{\Delta(t, \xi)} dt \leq c_2 \log \frac{1}{\varepsilon}.$$

To prove the  $C^\infty$  well-posedness of the Cauchy problem (1) in the reduced form (14), we first apply the Fourier transform in  $x$  and work on the equivalent system

$$\begin{aligned} D_t V - \mathcal{A}(t, \xi) V + \mathcal{L}(t, \xi) V &= 0, \\ V_{t=0} &= \{D_t^{j-1} \langle \xi \rangle^{m-j} \widehat{g}(\xi)\}_{j=1,2,\dots,m}, \end{aligned} \quad (23)$$

where  $V = \mathcal{F}_{x \rightarrow \xi} U(t, \cdot)(\xi)$ . We then consider the energy

$$E(t, \xi) = \begin{cases} |V(t, \xi)|^2 & \text{for } t \in A_{\xi/|\xi|, \varepsilon} \text{ and } \xi/|\xi| \in X, \\ \langle \mathcal{Q}(t, \xi) V(t, \xi), V(t, \xi) \rangle & \text{for } t \in [a, b] \setminus A_{\xi/|\xi|, \varepsilon} \text{ and } \xi/|\xi| \in X, \end{cases}$$

defined for  $t \in [0, T]$ ,  $\xi \in \mathbb{R}^n$  with  $\xi/|\xi| \in X$ , and  $\varepsilon \in (0, e^{-1}]$ . Note that  $\Delta(t, \xi) > 0$  when  $t \in [0, T] \setminus A_{\xi/|\xi|, \varepsilon}$  and  $\xi/|\xi| \in X$ , and, thanks to Lemma 4.4,  $[0, T] \setminus A_{\xi/|\xi|, \varepsilon}$  is a finite union of at most  $p$  closed intervals  $[c_i, d_i]$ . Moreover, the set  $A_{\xi/|\xi|, \varepsilon}$  is a finite union of open intervals whose total length does not exceed  $\varepsilon$ .

We now define a Kovalevskian energy on  $A_{\xi/|\xi|, \varepsilon}$  and a hyperbolic energy on the complement.

#### 4.1 The Kovalevskian energy

Let  $t \in [t', t''] \subset A_{\xi/|\xi|, \varepsilon}$  and  $\xi/|\xi| \in X$ . Hence

$$\begin{aligned} \partial_t E(t, \xi) &= 2\operatorname{Re}\langle V(t, \xi), \partial_t V(t, \xi) \rangle \\ &= 2\operatorname{Re}\langle V(t, \xi), i\langle \xi \rangle \mathcal{A}(t, \xi) V(t, \xi) + i\mathcal{L}(t, \xi) V(t, \xi) \rangle \leq 2(c_{\mathcal{A}}\langle \xi \rangle + c_{\mathcal{L}})E(t, \xi). \end{aligned}$$

By Gronwall's Lemma on  $[t', t'']$ , we get

$$|V(t, \xi)| \leq e^{(c_{\mathcal{A}}\langle \xi \rangle + c_{\mathcal{L}})(t-t')} |V(t', \xi)| \leq c e^{c\langle \xi \rangle(t-t')} |V(t', \xi)|. \quad (24)$$

#### 4.2 The hyperbolic energy

Let us work on any subinterval  $[c_i, d_i]$  of  $[0, T] \setminus A_{\xi/|\xi|, \varepsilon}$ . Assuming  $\xi/|\xi| \in X$ , we have that  $\Delta(t, \xi) > 0$  on  $[c_i, d_i]$ . By definition of the symmetriser, we have that

$$\begin{aligned} \partial_t E(t, \xi) &= \langle \partial_t \mathcal{Q}(t, \xi) V(t, \xi), V(t, \xi) \rangle \\ &\quad + \langle \mathcal{Q}(t, \xi) \partial_t V(t, \xi), V(t, \xi) \rangle + \langle \mathcal{Q}(t, \xi) V(t, \xi), \partial_t V(t, \xi) \rangle \\ &= \frac{\langle \partial_t \mathcal{Q}(t, \xi) V(t, \xi), V(t, \xi) \rangle}{\langle \mathcal{Q}(t, \xi) V, V \rangle} E(t, \xi) + \langle \mathcal{Q}(t, \xi) (i\langle \xi \rangle \mathcal{A}(t, \xi) \\ &\quad + i\mathcal{L}(t, \xi)) V(t, \xi), V(t, \xi) \rangle \\ &\quad + \langle \mathcal{Q}(t, \xi) V(t, \xi), (i\langle \xi \rangle \mathcal{A}(t, \xi) + i\mathcal{L}(t, \xi)) V(t, \xi) \rangle \\ &= \frac{\langle \partial_t \mathcal{Q}(t, \xi) V(t, \xi), V(t, \xi) \rangle}{\langle \mathcal{Q}(t, \xi) V, V \rangle} E(t, \xi) + i\langle (\mathcal{Q}(t, \xi) \mathcal{L}(t, \xi) \\ &\quad - \mathcal{L}^*(t, \xi) \mathcal{Q}(t, \xi)) V(t, \xi), V(t, \xi) \rangle. \end{aligned}$$

Now, by Lemma 4.1 and 4.2, the hypothesis (18) implies that the quantity

$$\frac{\langle \partial_t \mathcal{Q}(t, \xi) V(t, \xi), V(t, \xi) \rangle}{\langle \mathcal{Q}(t, \xi) V, V \rangle}$$

is bounded by

$$\sqrt{\frac{\tilde{\Delta}(t, \xi)}{\Delta(t, \xi)}}.$$

Hence, by definition of  $\tilde{\Delta}$  we conclude that

$$\begin{aligned} & \partial_t E(t, \xi) \\ & \leq C \sqrt{\frac{\tilde{\Delta}(t, \xi)}{\Delta(t, \xi)}} E(t, \xi) + |(\mathcal{Q}(t, \xi) \mathcal{L}(t, \xi) - \mathcal{L}^*(t, \xi) \mathcal{Q}(t, \xi)) V(t, \xi), V(t, \xi)| \\ & \leq C \left( 1 + \frac{|\partial_t \Delta(t, \xi)|}{\Delta(t, \xi)} \right) E(t, \xi) + |(\mathcal{Q}(t, \xi) \mathcal{L}(t, \xi) - \mathcal{L}^*(t, \xi) \mathcal{Q}(t, \xi)) V(t, \xi), V(t, \xi)|. \end{aligned} \quad (25)$$

We now have to deal with the lower-order terms. By arguing as in Remark 4.8 in [13] we can estimate

$$|(\mathcal{Q}(t, \xi) \mathcal{L}(t, \xi) - \mathcal{L}^*(t, \xi) \mathcal{Q}(t, \xi)) V(t, \xi), V(t, \xi)| \leq c \|\mathcal{L}\| \|V\|^2 + c \|\mathcal{L}^*\| \|V\|^2.$$

The hypothesis (19) combined with Proposition 3.2 implies that both  $\|\mathcal{L}\|$  and  $\|\mathcal{L}^*\|$  are bounded by

$$\Delta(t, \xi) + |\partial_t \Delta(t, \xi)|.$$

Hence, by applying Lemma 4.1 we arrive at the estimate

$$\begin{aligned} & |(\mathcal{Q}(t, \xi) \mathcal{L}(t, \xi) - \mathcal{L}^*(t, \xi) \mathcal{Q}(t, \xi)) V(t, \xi), V(t, \xi)| \\ & \leq C' \left( \frac{\Delta(t, \xi) + |\partial_t \Delta(t, \xi)|}{\Delta(t, \xi)} \right) E(t, \xi) \\ & \leq C' \left( 1 + \frac{|\partial_t \Delta(t, \xi)|}{\Delta(t, \xi)} \right) E(t, \xi). \end{aligned} \quad (26)$$

Finally, by combining (25) and (26) we obtain the final energy estimate

$$\partial_t E(t, \xi) \leq c' \left( 1 + \frac{|\partial_t \Delta(t, \xi)|}{\Delta(t, \xi)} \right) E(t, \xi). \quad (27)$$

### 4.3 Completion of the proof

We are now ready to prove Theorem 3.3.

*Proof of Theorem 3.3* We begin by observing that, by the finite speed of propagation for hyperbolic equations, we can always assume that the Cauchy data in (1) are compactly supported. We refer to the Kovalevskian energy and the hyperbolic energy introduced above. We note that in the energies under consideration we can assume  $|\xi| \geq 1$  since the continuity of

$V(t, \xi)$  in  $\xi$  implies that both energies are bounded for  $|\xi| \leq 1$ . Let us consider the hyperbolic energy on the interval  $[c_i, d_i]$ . By Gronwall's Lemma on  $[c_i, d_i]$ , we get the inequality

$$E(t, \xi) \leq e^{c(d_i - c_i)} \exp\left(c \int_{c_i}^t \frac{|\partial_s \Delta(s, \xi)|}{\Delta(s, \xi)} ds\right) E(c_i, \xi). \quad (28)$$

By Lemma 4.4, (ii), we have

$$\Delta(t, \xi) \geq \min_{s \in [a, b] \setminus A_{\xi, \varepsilon}} \Delta(s, \xi) \geq c_1 \varepsilon^{2q} \|\Delta(\cdot, \xi)\|_{L^\infty([a, b])},$$

for all  $t \in [c_i, d_i]$ . Hence, applying Lemma 4.1 to (28) we have that there exists a constant  $C > 0$  such that

$$\begin{aligned} |V(t, \xi)|^2 &\leq C \frac{1}{\varepsilon^{2q} \|\Delta(\cdot, \xi)\|_{L^\infty([a, b])}} \exp\left(\int_{c_i}^t \frac{|\partial_s \Delta(s, \xi)|}{\Delta(s, \xi)} ds\right) |V(c_i, \xi)|^2, \\ &\leq C \frac{1}{\varepsilon^{2q} \|\Delta(\cdot, \xi)\|_{L^\infty([a, b])}} e^{C \log(1/\varepsilon)} |V(c_i, \xi)|^2, \end{aligned} \quad (29)$$

for all  $t \in [c_i, d_i]$  and for  $|\xi| \geq 1$ . Note that in the estimate above we have used Lemma 4.4, (ii), in the last step. Since the number of the closed interval  $[c_i, d_i]$  does not exceed  $p$ , a combination of the Kovalevskian energy (24) with the hyperbolic energy (29) leads to

$$|V(b, \xi)| \leq C \frac{1}{\varepsilon^{pq} \|\Delta(\cdot, \xi)\|_{L^\infty([a, b])}^{p/2}} e^{C(\log(1/\varepsilon) + \varepsilon|\xi|)} |V(a, \xi)|,$$

for  $|\xi| \geq 1$ . At this point setting  $\varepsilon = e^{-1} \langle \xi \rangle^{-1}$  we have that there exist constants  $C' > 0$  and  $\kappa \in \mathbb{N}_0$  such that

$$|V(b, \xi)| \leq C' \langle \xi \rangle^{pq + \kappa} |V(a, \xi)|, \quad (30)$$

for  $|\xi| \geq 1$ . This proves the  $C^\infty$  well-posedness of the Cauchy problem (1). Similarly, (30) implies the well-posedness of (1) in  $\mathcal{D}'(\mathbb{R}^n)$ .  $\square$

**Open Access** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

## References

1. Bronshtein, M.D.: The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity. (Russian). *Trudy Moskov. Mat. Obshch.* **41**, 83–99 (1980)
2. Bronshtein, M.D.: The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity. (Russian). *Trans. Moscow Math. Soc.* **1**, 87–103 (1982)
3. Colombini, F., Kinoshita, T.: On the Gevrey well posedness of the Cauchy problem for weakly hyperbolic equations of higher order. *J. Differ. Equ.* **186**, 394–419 (2001)
4. Colombini, F., Kinoshita, T.: On the Gevrey wellposedness of the Cauchy problem for weakly hyperbolic equations of 4th order. *Hokkaido Math. J.* **31**, 39–60 (2002)
5. Colombini, F., Spagnolo, S.: An example of a weakly hyperbolic Cauchy problem not well posed in  $C^\infty$ . *Acta Math.* **148**, 243–253 (1982)
6. D'Ancona, P., Kinoshita, T.: On the wellposedness of the Cauchy problem for weakly hyperbolic equations of higher order. *Math. Nachr.* **278**, 1147–1162 (2005)

7. D'Ancona, P., Kinoshita, T., Spagnolo, S.: Weakly hyperbolic systems with Hölder continuous coefficients. *J. Differ. Equ.* **203**(1), 6481 (2004)
8. D'Ancona, P., Kinoshita, T., Spagnolo, S.: On the 2 by 2 weakly hyperbolic systems. *Osaka J. Math.* **45**(4), 921939 (2008)
9. D'Ancona, P., Spagnolo, S.: Quasi-symmetrization of hyperbolic systems and propagation of the analytic regularity. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8) **1**(1), 169–185 (1997)
10. Garetto, C.: On hyperbolic equations and systems with non-regular time dependent coefficients. *J. Differ. Equ.* **259**(11), 5846–5874 (2015)
11. Garetto, C., Ruzhansky, M.: On the well-posedness of weakly hyperbolic equations with time-dependent coefficients. *J. Differ. Equ.* **253**(5), 1317–1340 (2012)
12. Garetto, C., Ruzhansky, M.: Weakly hyperbolic equations with non-analytic coefficients and lower order terms. *Math. Ann.* **357**(2), 401–440 (2013)
13. Garetto, C., Ruzhansky, M.: A note on weakly hyperbolic equations with analytic principal part. *J. Math. Anal. Appl.* **412**(1), 1–14 (2014)
14. Garetto, C., Ruzhansky, M.: Hyperbolic second order equations with non-regular time dependent coefficients. *Arch. Ration. Mech. Anal.* **217**, 113–154 (2015)
15. Gramchev, T., Ruzhansky, M.: Cauchy problem for some  $2 \times 2$  hyperbolic systems of pseudo-differential equations with nondiagonalisable principal part. *Studies in phase space analysis with applications to PDEs. Progress in Nonlinear Differential Equations Applications*, vol. 84, pp. 129–145. Birkhäuser/Springer, New York (2013)
16. Ivrii, V.: Partial differential equations IV. *Encyclopedia of mathematics science* 20. In: Egorov, Y., Shubin, M. (eds.) *Linear Hyperbolic Equations*, pp. 149–235. Springer, Berlin (1993)
17. Jannelli, E.: The hyperbolic symmetrizer: theory and applications. *Advances in Phase Space Analysis of Partial Differential Equations*, vol. 78, pp. 113–139. Birkhäuser (2009)
18. Jannelli, E., Tagliabata, G.: Homogeneous weakly hyperbolic equations with time dependent analytic coefficients. *J. Differ. Equ.* **251**, 995–1029 (2011)
19. Kajitani, K.: Local solutions of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes. *Hokkaido Math. J.* **12**(3 part 2), 434–460 (1983)
20. Kajitani, K., Yuzawa, Y.: The Cauchy problem for hyperbolic systems with Hölder continuous coefficients with respect to the time variable. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5**(4), 465–482 (2006)
21. Kamotski, I., Ruzhansky, M.: Regularity properties, representation of solutions and spectral asymptotics of systems with multiplicities. *Commun. Partial Differ. Equ.* **32**, 1–35 (2007)
22. Kinoshita, T., Spagnolo, S.: Hyperbolic equations with non-analytic coefficients. *Math. Ann.* **336**, 551–569 (2006)
23. Nishitani, T.: Cauchy problem for noneffectively hyperbolic operators. *MSJ Memoirs*, 30. Mathematical Society of Japan, Tokyo, (2013)
24. Nishitani, T.: Hyperbolic systems with analytic coefficients. Well-posedness of the Cauchy problem. *Lecture Notes in Mathematics*, 2097. Springer, Cham, (2014)
25. Nishitani, T.: On the Cauchy problem for hyperbolic operators with double characteristics, a transition case. *Fourier analysis, Trends Mathematics*, pp. 311–334. Birkhäuser/Springer, Cham, (2014)
26. Ruzhansky, M., Tokmagambetov, N.: Very weak solutions of wave equation for Landau Hamiltonian with irregular electromagnetic field. *Lett. Math. Phys.* (2016). doi:[10.1007/s11005-016-0919-6](https://doi.org/10.1007/s11005-016-0919-6)
27. Ruzhansky, M., Wirth, J.: Dispersive estimates for hyperbolic systems with time-dependent coefficients. *J. Differ. Equ.* **251**, 941–969 (2011)
28. Ruzhansky, M., Wirth, J.: Asymptotic behaviour of solutions to hyperbolic equations and systems. *Variable Lebesgue spaces and hyperbolic systems, Adv. Courses Math. CRM Barcelona*, pp. 91–169, Birkhäuser/Springer, Basel (2014)
29. Yuzawa, Y.: The Cauchy problem for hyperbolic systems with Hölder continuous coefficients with respect to time. *J. Differ. Equ.* **219**(2), 363–374 (2005)